

CIII. The Invention of a General Method for determining the Sum of every 2d, 3d, 4th, or 5th, &c. Term of a Series, taken in order; the Sum of the whole Series being known. By Thomas Simpson, F.R.S.

Read Nov. 16,

1758.

AS the doctrine of Series' is of very great use in the higher branches of the mathematics, and their application to nature, every attempt tending to extend that doctrine may justly merit some degree of regard. The subject of the paper, which I have now the honour to lay before the Society, will be found an improvement of some consequence in that part of science. And how far the business of finding fluents may, in some cases, be facilitated thereby, will appear from the examples subjoined, in illustration of the general method here delivered.

The series propounded, whose sum (S) is supposed to be given (either in algebraic terms, or by the measures of angles and ratio's, &c.) I shall here represent by $a + bx + cx^2 + dx^3 + ex^4$, &c. and shall first give the solution of that case, where every third term is required to be taken, or where the series to be summed is $a + dx^3 + gx^6 + kx^9$, &c. By means whereof, the general method of proceeding, and the resolution of every other case, will appear evident.

Here, then, every *third* term being required to be taken, let the series ($a + dx^3 + gx^6$, &c.), whose value

value is sought, be conceived to be composed of three others.

$$\frac{1}{2} \times a + b \times px + c \times px^2 + d \times px^3 + e \times px^4, \text{ &c.}$$

$$\frac{1}{2} \times a + b \times qx + c \times qx^2 + d \times qx^3 + e \times qx^4, \text{ &c.}$$

$$\frac{1}{2} \times a + b \times rx + c \times rx^2 + d \times rx^3 + e \times rx^4, \text{ &c.}$$

having all the *same form*, and the *same coefficients* with the series first proposed, and wherein the converging quantities px , qx , rx , are also in a determinate (tho' yet unknown) ratio to the original converging quantity x . Now, in order to determine the quantities of these ratios, or the values of p , q , and r , let the terms containing the same powers of x , in the two equal values, be equated in the common way:

So shall,

$$\frac{1}{2}b \times px + \frac{1}{3}b \times qx + \frac{1}{3}b \times rx = 0$$

$$\frac{1}{3}c \times p^2x^2 + \frac{1}{2}c \times q^2x^2 + \frac{1}{3}c \times r^2x^2 = 0$$

$$\frac{1}{3}d \times p^3x^3 + \frac{1}{3}d \times q^3x^3 + \frac{1}{3}d \times r^3x^3 = dx^3$$

$$\frac{1}{3}e \times p^4x^4 + \frac{1}{3}e \times q^4x^4 + \frac{1}{3}e \times r^4x^4 = 0$$

&c.

And consequently,

$$p + q + r = 0$$

$$p^2 + q^2 + r^2 = 0$$

$$p^3 + q^3 + r^3 = 3$$

$$p^4 + q^4 + r^4 = 0, \text{ &c.}$$

Make, now, $p^3 = 1$, $q^3 = 1$, and $r^3 = 1$; that is, let p , q , and r , be the three roots of the cubic equation $x^3 = 1$, or $x^3 - 1 = 0$: then, seeing both the second and third terms of this equation are wanting,

not

not only the sum of all the roots ($p + q + r$) but the sum of all their squares ($p^2 + q^2 + r^2$) will vanish, or be equal to nothing (by common algebra), as they ought, to fulfil the conditions of the two first equations. Moreover, since $p^3 = 1$, $q^3 = 1$, and $r^3 = 1$, it is also evident, that $p^4 + q^4 + r^4 (= p + q + r) = 0$, $p^5 + q^5 + r^5 (= p^2 + q^2 + r^2) = 0$, $p^6 + q^6 + r^6 (= p^3 + q^3 + r^3) = 3$. Which equations being, in effect, nothing more than the first three repeated, the values of p , q , r , above assigned, equally fulfil the conditions of these also: so that the series arising from the addition of three assumed ones will agree, in every term, with *that* whose sum is required: but those series' (whereof the quantity in question is composed) having all of them the *same form* and the *same coefficients* with the original series $a + bx + cx^2 + dx^3$, &c. ($= S$), their sums will therefore be truly obtained, by substituting px , qx , and rx , successively, for x , in the given value of S . And, by the very same reasoning, and the process above laid down, it is evident, that, if every n^{th} term (instead of every third term) of the given series be taken, the values of p , q , r , s , &c. will then be the roots of the equation $x^n - 1 = 0$ *; and that, the

* If α , β , γ , δ , &c. be supposed to represent the co-sines of the angles $\frac{360^\circ}{n}$, $2 \times \frac{360^\circ}{n}$, $3 \times \frac{360^\circ}{n}$, &c. (the radius being unity); then the roots of the equation $x^n - 1 = 0$ (expressing the several values of p , q , r , s , &c.) will be truly defined by 1 , $\alpha + \sqrt{\alpha\alpha - 1}$, $\alpha - \sqrt{\alpha\alpha - 1}$, $\beta + \sqrt{\beta\beta - 1}$, $\beta - \sqrt{\beta\beta - 1}$, &c. The demonstration of this will be given farther on.

sum of all the terms so taken, will be truly obtained by substituting px , qx , rx , sx , &c. successively for x , in the given value of S , and then dividing the sum of all the quantities thence arising by the given number n .

The same method of solution holds equally, when, in taking every n^{th} term of the series, the operation begins at some term after the first. For all the terms preceding *that* may be transposed, and the whole equation divided by the power of x in the first of the remaining terms; and then the sum of every n^{th} term (beginning at the first) will be found by the preceding directions; which sum, multiplied by the power of x that before divided, will evidently give the true value required to be determined. Thus, for example, let it be required to find the sum of every third term of the given series $a + bx + cx^2 + dx^3 + ex^4$, &c. ($= S$), beginning with cx^2 . Then, by transposing the two first terms, and dividing the whole by x^2 , we shall have $c + dx + ex^2 + fx^3$, &c. $= \frac{S - a - bx}{x^2}$ ($= S'$). From whence having found the sum of every third term of the series $c + dx + ex^2 + fx^3$, &c. beginning at the first (c), that sum, multiplied by x^2 , will manifestly give the true value sought in the present case.

And here it may be worth while to observe, that all the terms preceding *that* at which the operation (in any case) begins, may (provided they exceed not in number the given interval n) be intirely disregarded,

regarded, as having no effect at all in the result.

For if in that part $\left(\frac{-a - bx}{xx} \right)$ of the value of S' , above exhibited, in which the first terms, a and bx , enter, there be substituted px , qx , rx , successively, for x (according to the *prescript*) the sum of the quantities thence arising will be

$$\begin{aligned} & -\frac{a}{p^2x^2} - \frac{a}{q^2x^2} - \frac{a}{r^2x^2} \\ & -\frac{b}{px} - \frac{b}{qx} - \frac{b}{rx} \end{aligned}$$

which, because $p^3 = 1$, $q^3 = 1$, &c. (or $p^2 = \frac{1}{p}$, $q^2 = \frac{1}{q}$, &c.) may be expressed thus;

$$\begin{aligned} & -\frac{a}{xx} \times \overline{p + q + r} \\ & -\frac{b}{x} \times \overline{p^2 + q^2 + r^2} \end{aligned}$$

But, that $p + q + r = 0$, and $p^2 + q^2 + r^2 = 0$, hath been already shewn ; whence the truth of the general observation is manifest. Hence it also appears, that the method of solution above delivered, is not only general, but includes this singular beauty and advantage, that in all series' whatever, whereof the terms are to be taken according to the same assigned order, the quantities (p , q , r , &c.), whereby the resolution is performed, will remain invariably the same. The greater part of these quantities are indeed *imaginary* ones ; and so likewise will the quantities be that result from them, when substitution is made in the given expression for the value of S . But by adding, as is usual in like cases, every two corresponding va-

lues, so resulting together, all marks of *impossibility* will disappear.

If, in the series to be summed, the alternate terms (*viz.* the 2d, 4th, 6th, &c.) should be required to be taken under signs contrary to what they have in the original series given; the reasoning and result will be no-ways different; only, instead of making $p^3 + q^3 + r^3$ (or $p^n + q^n + r^n$, &c.) = + 3 (or + n), the same quantity must, here, be made = — 3 (or — n). From whence, p^n being = — 1, q^n = — 1, &c. the values of p , q , r , &c. will, in this case, be the roots of the equation $x^n + 1 = 0$.

It may be proper, now, to put down an example, or two, of the use and application of the general conclusions above derived. First, then, supposing

the series, whose sum is given, to be $x + \frac{x^2}{2} +$

$$\frac{x^3}{3} + \frac{x^4}{4} \dots \dots + \frac{x^m}{m} + \frac{x^{m+1}}{m+1} + \frac{x^{m+2}}{m+2} \dots$$

$$+ \frac{x^{m+n}}{m+n} + \frac{x^{m+n+1}}{m+n+1} +, \text{ &c.} = - H. \text{ Log.}$$

$\frac{1-x}{1-x}$ (= S); let it be required, from hence, to find the sum of the series $\left(\frac{x^m}{m} + \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} \right)$ &c.) arising by taking every n^{th} term thereof, beginning with that whose exponent (m) is any integer less than n . Here, the terms preceding $\frac{x^m}{m}$ being transposed, and the whole equation divided by x^n ; we

[763]

we shall have $\frac{1}{m} + \frac{x}{m+1} + \frac{x^2}{m+2} + \frac{x^3}{m+3}$, &c.

$= - \frac{1}{x^m} \times H. \text{Log. } \frac{1-x}{1+x} - \frac{x + \frac{1}{2}x^2}{x^m}, \text{ &c.}$. In

which value, let $p x, q x, r x$, &c. be, successively, substituted for x (according to prescript) neglecting intirely the terms $\frac{x + \frac{1}{2}x^2}{x^m}$, as having no effect at all

in the result : from whence we get $- \frac{1}{px^n} \times \text{Log. } \frac{1}{1-px}$

$\frac{1}{1-qx} = \frac{1}{qx^n} \times \text{Log. } \frac{1}{1-qx} = \frac{1}{rx^n} \times \text{Log. }$

$\frac{1}{1-rx}$, &c. Which multiplied by x^n (the quantity that before divided) gives $- \frac{1}{p^n} \times \text{Log. } \frac{1}{1-px} -$

$\frac{1}{q^n} \times \text{Log. } \frac{1}{1-qx} - \frac{1}{r^n} \times \text{Log. } \frac{1}{1-rx}$, &c. =

n times the quantity required to be determined.

But now, to get rid of the imaginary quantities q , r , &c. by means of their known values $\alpha + \sqrt{\alpha\alpha - 1}$, $\alpha - \sqrt{\alpha\alpha - 1}$, &c. it will be necessary to observe, that, as the product of any two corresponding ones ($\alpha + \sqrt{\alpha\alpha - 1} \times \alpha - \sqrt{\alpha\alpha - 1}$) is equal to unity, we may therefore write $\alpha - \sqrt{\alpha\alpha - 1}^n (= r^n)$ instead of its equal $\frac{1}{q^n}$, and $\alpha + \sqrt{\alpha\alpha - 1}^n (= q^n)$

instead of its equal $\frac{1}{r^n}$: by which means the two

terms, wherein these two quantities enter, will stand thus; $-\alpha - \sqrt{\alpha\alpha - 1}'' \times \text{Log. } \frac{1 - qx}{1 + \sqrt{\alpha\alpha - 1}''}$ $\times \text{Log. } \frac{1 - rx}{1 + \sqrt{\alpha\alpha - 1}''}$.

But, if A be assumed to express the co-sine of an arch (\mathcal{Q}), m times as great as that $(\frac{360^\circ}{n})$ whose co-sine is here denoted by α ; then will $A - \sqrt{AA - 1} = * \alpha - \sqrt{\alpha\alpha - 1}''$, and $A + \sqrt{AA - 1} =$

* Because $\frac{-x^*}{\sqrt{1 - xx}}$ and $\frac{-X^*}{\sqrt{1 - XX}}$ are known to express the fluxions of the circular arcs whose co-sines are x and X , it is evident, if those arcs be supposed in any constant ratio of 1 to n , that $\frac{nx^*}{\sqrt{1 - xx}} = \frac{X^*}{\sqrt{1 - XX}}$, and consequently that $\frac{nx^*}{\sqrt{xx - 1}} = (\frac{nx^*}{\sqrt{-1 \times \sqrt{1 - xx}}} = \frac{X^*}{\sqrt{-1 \times \sqrt{1 - XX}}}) = \frac{X^*}{\sqrt{XX - 1}}$.

From whence, by taking the fluents, $n \times \text{Log. } x + \sqrt{xx - 1}$ (or $\text{Log. } x + \sqrt{xx - 1}''$) = $\text{Log. } X + \sqrt{XX - 1}$; and consequently $x + \sqrt{xx - 1}'' = X + \sqrt{XX - 1}$: whence also, seeing $x - \sqrt{xx - 1}$ is the reciprocal of $x + \sqrt{xx - 1}$, and $X - \sqrt{XX - 1}$ of $X + \sqrt{XX - 1}$, it is likewise evident, that $x - \sqrt{xx - 1}'' = X - \sqrt{XX - 1}$. Hence, not only the truth of the above assumption, but what has been advanced in relation to the roots of the equation $z^n - 1 = 0$, will appear manifest. For if $x \pm \sqrt{xx - 1}$ be put = z , then will $z^n (= x \pm \sqrt{xx - 1}'') = X \pm \sqrt{XX - 1}$: where, assuming $X = 1 = \text{co-s. } 0 = \text{co-s. } 360^\circ = \text{co-s. } 2 \times 360^\circ = \text{co-s. } 3 \times 360^\circ$, &c. the equation will become $z^n = 1$, or $z^n - 1 = 0$; and the different values of x , in the expression $(x \pm \sqrt{xx - 1})$ for the root z , will consequently be the co-sines of the arcs, $\frac{o}{n} \frac{360^\circ}{n}$, $\frac{2 \times 360^\circ}{n}$, &c. these arcs being

the

$\alpha + \sqrt{\alpha\alpha - 1}$: which values being substituted above, we thence get

$$-A \times \log. 1 - qx + \log. 1 - rx \\ + \sqrt{AA - 1} \times \log. 1 - qx - \log. 1 - rx;$$

whereof the former part (which, exclusive of the factor A , I shall hereafter denote by M) is manifestly equal to $-A \times \log. 1 - qx \times 1 - rx$ (by the nature of logarithms) $= -A \times \log. 1 - q + rx + qr x^2 = -A \times \log. 1 - 2\alpha x + xx$ (by substituting the values of q and r): which is now intirely free from imaginary quantities. But, in order to exterminate them out of the latter part also, put $y = \log. 1 - qx - \log. 1 - rx$; then will $\dot{y} = \frac{-qx}{1 - qx}$
 $+ \frac{rx}{1 - rx} = - \frac{q - r \times \dot{x}}{1 - q + rx + xx} = - \frac{2\sqrt{\alpha\alpha - 1} \times \dot{x}}{1 - 2\alpha x + xx}$
 $= - \frac{2\sqrt{-1} \times \sqrt{1 - \alpha\alpha} \times \dot{x}}{1 - 2\alpha x + xx}$; where $\frac{\sqrt{1 - \alpha\alpha} \times \dot{x}}{1 - 2\alpha x + xx}$ expresseth the fluxion of a circular arch (N) whose radius is 1 , and sine $= \frac{\sqrt{1 - \alpha\alpha} \times \dot{x}}{1 - 2\alpha x + xx}$; consequently y will be $= -2\sqrt{-1} \times N$: which, multiplied by $\sqrt{AA - 1}$, or its equal $\sqrt{-1} \times \sqrt{1 - AA}$, gives $2\sqrt{1 - AA} \times N$;

the corresponding *submultiples* of those above, answering to the co-sine $X (= 1)$. — In the same manner, if X be taken $= -1 = \text{co-f. } 180^\circ = \text{co-f. } 3 \times 180^\circ = \text{co-f. } 5 \times 180^\circ, \&c.$ then will $x^n = -1$, or $x^n + 1 = 0$; and the values of x will, in this case, be the co-sines of $\frac{180^\circ}{n}$, $3 \times \frac{180^\circ}{n}$, $5 \times \frac{180^\circ}{n}$, &c.

and,

and, this value being added to that of the former part (found above), and the whole being divided by n , we thence obtain $\frac{-AM + 2\sqrt{1 - AA} \times N}{n}$, or $\frac{1}{n} \times \overline{-\text{co-s. } Q \times M + \text{fin. } Q \times 2N}$ for that part of the value sought depending on the two terms affected with q and r . From whence the sum of any other two corresponding terms will be had, by barely substituting one letter, or value, for another: So that,

$$\frac{1}{n} \times \begin{cases} -\log. \overline{1 - x} \\ -\text{co-s. } Q \times M + \text{fin. } Q \times 2N \\ -\text{co-s. } Q' \times M' + \text{fin. } Q' \times 2N' \\ -\text{co-s. } Q'' \times M'' + \text{fin. } Q'' \times 2N'' \\ \text{-- &c.} \quad \quad \quad + \text{&c.} \end{cases}$$

will truly express the sum of the series proposed to be determined; $M, M', M'' \&c.$ being the hyperbolical logarithms of $1 - 2\alpha x + xx, 1 - 2\beta x + xx, 1 - 2\gamma x + xx, \&c.$ $N, N', N'' \&c.$ the arcs whose fines are $\frac{x\sqrt{1 - \alpha\alpha}}{\sqrt{1 - 2\alpha x + xx}}, \frac{x\sqrt{1 - \beta\beta}}{\sqrt{1 - 2\beta x + xx}}, \frac{x\sqrt{1 - \gamma\gamma}}{\sqrt{1 - 2\gamma x + xx}}, \&c.$ and $Q, Q', Q'' \&c.$ the measures of the angles expressed by $\frac{360^\circ}{n} \times m, 2 \times \frac{360^\circ}{n} \times m, 3 \times \frac{360^\circ}{n} \times m, \&c.$ And here it may not be amiss to take

notice, that the series $\frac{x^m}{m} + \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} + \&c.$ thus determined, is that expressing the fluent of $\frac{x^{m-1}x}{1-x^n}$; corresponding to one of the two famous

Cotesian forms. From whence, and the reasoning above laid down, the fluent of the other form, $\frac{x^m - 1 \dot{x}}{1 + x^n}$, may be very readily deduced. For, since the series $\left(\frac{x^m}{m} - \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} - \frac{x^{m+3n}}{m+3n} \right)$ &c.) for this last fluent, is that which arises by changing the signs of the alternate terms of the former; the quantities $p, q, r, \&c.$ will here (agreeably to a preceding observation) be the roots of the equation $x^n + 1 = 0$; and, consequently, $\alpha, \beta, \gamma, \delta, \&c.$ the co-sines of the arcs $\frac{180^\circ}{n}, 3 \times \frac{180^\circ}{n}, 5 \times \frac{180^\circ}{n}, \&c.$ (as appears by the foregoing note). So that, making $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'', \&c.$ equal, here, to the measures of the angles $\frac{180^\circ}{n} \times m, 3 \times \frac{180^\circ}{n} \times m, 5 \times \frac{180^\circ}{n} \times m, \&c.$ the fluent sought will be expressed in the very same manner as in the preceding case; except that the first term, $-\log. 1 - x$ (arising from the *rational* root $p = 1$) will here have no place.

After the same manner, with a small increase of trouble, the fluent of $\frac{x^m - 1 \dot{x}}{1 \pm 2 \sqrt{x^n + x^{2n}}}$ may be derived, m and n being any integers whatever. But I shall now put down one example, wherein the impossible quantities become exponents of the powers, in the terms where they are concerned.

The series here given is $1 - x + \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} - \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5}, \&c.$ = the number whose hyp. log. is

is $-x$, and it is required to find the sum of every n^{th} term thereof, beginning at the first. Here the quantity sought will (according to the general rule) be truly defined by the n^{th} part of the sum of all the numbers whose respective logarithms are $-px$, $-qx$, $-rx$, &c.; which numbers, if N be taken to denote the number whose hyp. log. = 1, will be truly expressed by N^{-px} , N^{-qx} , N^{-rx} , &c. From whence, by writing for p, q, r , &c. their equals $1, \alpha + \sqrt{\alpha\alpha - 1}, \alpha - \sqrt{\alpha\alpha - 1}, \beta + \sqrt{\beta\beta - 1}, \beta - \sqrt{\beta\beta - 1}$, &c. and putting $\alpha' = \sqrt{1 - \alpha\alpha}$, $\beta' = \sqrt{1 - \beta\beta}$, &c. we shall have $\frac{1}{n} \times N^{-px} + N^{-qx} + N^{-rx}$, &c. = $\frac{1}{n}$ into $N^{-x} + N^{-\alpha'x} \times N^{-\alpha x \sqrt{-1}} + N^{-\alpha x \sqrt{-1}} + N^{-\beta'x} \times N^{-\beta x \sqrt{-1}} + N^{\beta'x} \sqrt{-1} + \dots$. But $N^{-\alpha x \sqrt{-1}} + N^{-\beta x \sqrt{-1}}$ is known to express the double of the co-sine of the arch whose measure (to the radius 1) is $\alpha'x$. Therefore we have $\frac{1}{n}$ into $N^{-x} + N^{-\alpha'x} \times 2 \text{ co-s. } \alpha'x + N^{-\beta'x} \times 2 \text{ co-s. } \beta'x$, &c. for the true sum, or value proposed to be determined.

The solution of this case, in a manner a little different, I have given some time since, in another place; where the principles of the general method, here extended and illustrated, are pointed out. I shall put an end to this paper with observing, that if, in the series

series given, the even powers of x , or any other terms whatever, be wanting, their places must be supplied with cyphers ; which, in order the of numbering off, must be reckoned as real terms.

CIV. *Observatio Eclipsis Lunæ Die 30 Julii 1757. habita Olissipone à Joanne Chevalier, Congregationis Oratorii Presbytero, é Regia Londinensi Societate. Communicated by Jacob de Castro Sarmiento, M D F. R. S.*

Tubo optico 8 pedum.

		h			
Read Nov. 16. 1758.	I Nitium penumbræ	9	15	18	
	Initium dubium eclipsis	9	22	24	
Certo jam incœperat	—	9	23	34	
Umbra ad mare humorum observata vitro plano cœruleo	—	9	31	2	
Solo tubo optico observata	—	9	31	29	
Vitro flavo observata	—	9	31	48	
Umbra tangit Grimaldum observata vitro plano cœruleo	—	9	31	20	
Solo tubo optico	—	9	31	50	
Vitro plano flavo	—	9	32	8	
Totus Grimaldus tegitur observatus vitro plano cœruleo	—	9	34	4	
Solo tubo optico	—	9	34	28	
Vitro flavo	—	9	34	47	
VOL. 50.	5 F				Umbra